Math 3B — Week 2

Areas between curves Compute the areas of the regions bounded by

(a) $y = x$ and $y = x^2$.

(b) $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$, and $x = 5$.

(c) $y = x$, $y = \sin(x)$, $x = 0$, and $x = \pi$.

(d) $f(x) = x(x + 3)(x - 3)$ and $g(x) = 0$.

(e) $f(x) = x$, $g(x) = 2x$, and $h(x) = 5 - x$.

(f) $f(y) = \frac{1}{2}y^2 - 3$ and $g(x) = x - 1$.

The first two of these problems is done without appealing to a graph. The second one in particular is demonstrating that even if you don't know what the graphs look like, you can still find the area! I would still recommend graphing the functions yourself and seeing how the picture fits with the solution!

(a) The first step for all of these will be to find the points of intersection. In this case, we let $x = x^2$ and conclude that the intersections occur at $x = 0$ and $x = 1$. Since $x > x^2$ in this domain, we compute the area as

$$\begin{aligned}\int_0^1 x - x^2\,dx &= \frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_0^1 \\ &= \frac{1}{2} - \frac{1}{3} - 0 \\ &= \frac{1}{6}.\end{aligned}$$

(b) This problem is more complicated than the previous one, and there are several reasonable ways to approach it. Here is one. The problem has not done it for us, so let's begin by naming our functions $f(x) = 2x^2 + 10$ and $g(x) = 4x + 16$ so that we can refer to them more easily. The area we wish to find is given by the integral

$$\int_{-2}^{5} |f(x) - g(x)|\,dx.$$

To compute this integral, we need to figure out where $f(x) > g(x)$ and where $g(x) > f(x)$. Every time this changes, the functions will cross each other (since they're both continuous), so it will be helpful to find where the two functions intersect. As in part (a), we set $2x^2+10 = 4x + 16$, or

$$2x^2 - 4x - 6 = 0.$$

We could now use the quadratic formula, or we could notice that $2x^2-4x-6 = (2x+2)(x-3)$. Either way, we find intersections at $x = -1$ and $x = 3$. So, we know that the two functions do not cross each other between -2 and -1, between -1 and 3, and between 3 and 5, but that they may cross each other at -1 and 3. With this in mind, we split our integral into

$$\int_{-2}^{5} |f(x) - g(x)|\,dx = \int_{-2}^{-1} |f(x) - g(x)|\,dx + \int_{-1}^{3} |f(x) - g(x)|\,dx + \int_{3}^{5} |f(x) - g(x)|\,dx.$$

Now (and this is an important logical step!), since $f(x)$ and $g(x)$ do not cross each other inside any of the three intervals we are integrating over, we can move the absolute value out of the integral:

$$\int_{-2}^{5} |f(x) - g(x)|\, dx = \left|\int_{-2}^{-1} f(x) - g(x)\, dx\right| + \left|\int_{-1}^{3} f(x) - g(x)\, dx\right| + \left|\int_{3}^{5} f(x) - g(x)\, dx\right|.$$

If it's not clear why we can do this, you should stop and think about it! And now, we can just compute all of these. Since $f(x) - g(x) = 2x^2 - 4x - 6$, we have

$$\left|\int_{-2}^{-1} f(x) - g(x)\, dx\right| = \left|\frac{2}{3}x^3 - 2x^2 - 6x\Big|_{-2}^{-1}\right| = \left|\frac{10}{3} - \left(-\frac{4}{3}\right)\right| = \frac{14}{3},$$
$$\left|\int_{-1}^{3} f(x) - g(x)\, dx\right| = \left|\frac{2}{3}x^3 - 2x^2 - 6x\Big|_{-1}^{3}\right| = \left|-18 - \frac{10}{3}\right| = \frac{64}{3},$$
$$\left|\int_{3}^{5} f(x) - g(x)\, dx\right| = \left|\frac{2}{3}x^3 - 2x^2 - 6x\Big|_{3}^{5}\right| = \left|\frac{10}{3} - (-18)\right| = \frac{64}{3}.$$

While it's not necessary for solving the problem, note that whether the integral is positive or negative before taking the absolute value tells us whether $f(x) > g(x)$ or $g(x) > f(x)$! Finally, we find our total area to be

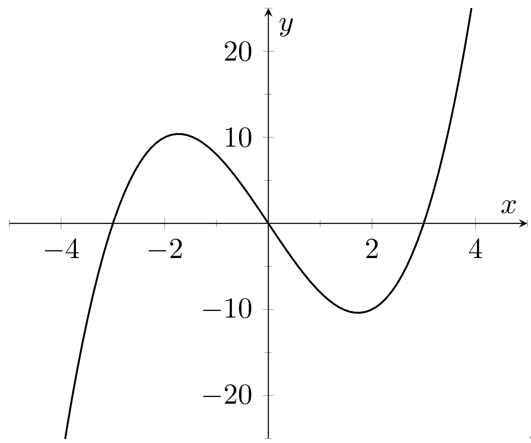
$$\int_{-2}^{5} |f(x) - g(x)|\, dx = \frac{14}{3} + \frac{64}{3} + \frac{64}{3} = \frac{142}{3}.$$

(c) Here, we are finding an area from 0 to π. We know roughly what x and $\sin(x)$ look like and that $\sin(0) = 0$, but we need to know if there are any other intersection points between 0 and π. To do this, we can appeal to our knowledge of derivatives. We know $\frac{d}{dx}\sin(x) = \cos(x)$, so the slope of $\sin(x)$ is 1 at $x = 0$. This equals the slope of $y = x$ at $x = 0$. Between 0 and π, we know $\cos(x) < 1$, so the function $y = \sin(x)$ increases no faster than the function $y = x$. Thus, there can be no more points of intersection, and we know $x \geq \sin(x)$. Hence, we can find the desired area as

$$\begin{aligned}\int_0^{\pi} x - \sin(x)\, dx &= \frac{1}{2}x^2 + \cos(x)\Big|_0^{\pi} \\ &= \left(\frac{1}{2}\pi^2 + \cos(\pi)\right) - (0 + \cos(0)) \\ &= \frac{1}{2}\pi^2 - 2.\end{aligned}$$

(d) This problem is asking us to find the area bounded by the function $f(x)$ and the x-axis. This is effectively the integral of $f(x)$, except that we do not want a signed integral. What we are asking for is the integral of $|f(x)|$.

Upon first inspection, it may seem inconvenient that f is presented as a product of terms, but it is actually very useful! We see immediately that f is the following cubic polynomial with roots at 0, -3, and 3:



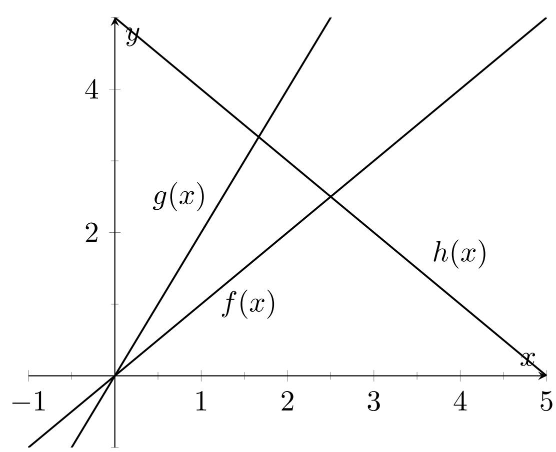
Of course, we can also verify that f is an odd function by noting that

$$\begin{aligned} f(-a) &= (-a)(-a+3)(-a-3) \\ &= -a\left[-(a-3)\right]\left[-(a+3)\right] \\ &= -a(a-3)(a+3) \\ &= -f(a). \end{aligned}$$

Now, we just compute the desired area as

$$\begin{aligned} -2\int_0^3 x(x-3)(x+3)\,dx &= -2\int_0^3 x^3 - 9x\,dx \\ &= -2\left(\frac{1}{4}x^4 - \frac{9}{2}x^2\right)\Bigg|_0^3 \\ &= -2\left[\left(\frac{81}{4} - \frac{81}{2}\right) - 0\right] \\ &= \frac{81}{2}. \end{aligned}$$

(e) This problem can be done without first drawing the lines to visualise what is going on (and it may be instructive to think about how to do that!), but for ease of explanation, let us just draw the plots and appeal to them for the solution.



From this plot, we can see that finding this area will require two integrals, perhaps one from the intersection of f and g to the intersection of g and h and one from the intersection of g and h to the intersection of f and h. Let us find all of these intersections. First, we set

$$x = 2x$$

to find that f and g intersect at $x = 0$. Next, we set

$$2x = 5 - x$$

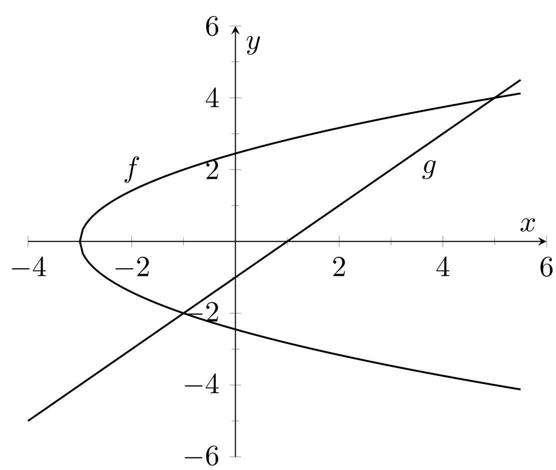
to find that g and h intersect at $x = 5/3$. Finally, we set

$$x = 5 - x$$

to find that f and h intersect at $x = 5/2$. Hence, the area we wish to find is given by

$$\begin{aligned} \int_0^{5/3} g(x) - f(x)\,dx + \int_{5/3}^{5/2} h(x) - f(x)\,dx &= \int_0^{5/3} 2x - x\,dx + \int_{5/3}^{5/2} 5 - x - x\,dx \\ &= \int_0^{5/3} x\,dx + \int_{5/3}^{5/2} 5 - 2x\,dx \\ &= \frac{1}{2}x^2\Big|_0^{5/3} \quad + \quad 5x - x^2\Big|_{5/3}^{5/2} \\ &= \left(\frac{25}{18} - 0\right) + \left(\frac{25}{4} - \frac{50}{9}\right) \\ &= \frac{25}{12}. \end{aligned}$$

(f) This problem is quite instructive, and we should begin with a plot of the two functions. Note that f is defined in terms of y, but we can plot it all the same!



Now, there are two ways we may go about finding the area of this region. The first is to split it into two integrals where the graphs intersect the first time. The other is to integrate with respect to y and avoid having to use two integrals! The latter is the approach we shall take, though the former is perhaps a useful exercise (what functions go in the integral from $x = -3$ to $x = -1$? You'll have to solve $x = f(y)$ for y, remembering that there is a $\pm$).

To integrate with respect to y, we need to find the y-values at which the two functions intersect. To do so, we write the function g as $x = y + 1$, set the x-values equal to each other, and find

$$\frac{1}{2}y^2 - 3 = y + 1$$

or

$$y^2 - 2y - 8 = 0.$$

This equation gives us $y = -2, 4$, so we find an area of

$$\begin{aligned}
\int_{-2}^{4} g(y) - f(y)\,dy &= \int_{-2}^{4} y + 1 - \left(\frac{1}{2}y^2 - 3\right)\,dy \\
&= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\,dy \\
&= -\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\Big|_{-2}^{4} \\
&= \left(-\frac{64}{6} + \frac{16}{2} + 16\right) - \left(\frac{8}{6} + \frac{4}{2} - 8\right) \\
&= 18.
\end{aligned}$$